Mean value surfaces with prescribed curvature form

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Abstract: The Gaussian curvature of a two-dimensional Riemannian manifold is uniquely determined by the choice of the metric. The formulas for computing the curvature in terms of components of the metric, in isothermal coordinates, involve the Laplacian operator and therefore, the problem of finding a Riemannian metric for a given curvature form may be viewed as a potential theory problem. This problem has, generally speaking, a multitude of solutions. To specify the solution uniquely, we ask that the metric have the mean value property for harmonic functions with respect to some given point. This means that we assume that the surface is simply connected and that it has a smooth boundary. In terms of the so-called metric potential, we are looking for a unique smooth solution to a nonlinear fourth order elliptic partial differential equation with second order Cauchy data given on the boundary. We find a simple condition on the curvature form which ensures that there exists a smooth mean value surface solution. It reads: the curvature form plus half the curvature form for the hyperbolic plane (with the same coordinates) should be ≤ 0. The same analysis leads to results on the question of whether the canonical divisors in weighted Bergman spaces over the unit disk have extraneous zeros. Numerical work suggests that the above condition on the curvature form is essentially sharp.

Our problem is in spirit analogous to the classical Minkowski problem, where the sphere supplies the chart coordinates via the Gauss map.

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1 Introduction

General abstract surfaces. Consider a simply connected C^∞-smooth bordered two-dimensional Riemannian manifold $\mathbf{\Omega}$. The boundary $\partial\mathbf{\Omega}$ is then a

C^∞-smooth Jordan curve. We model $\mathbf{\Omega}$ as the unit disk $\mathbb{D}$, supplied with the C^∞-smooth Riemannian metric $d\mathbf{s}$:

$$\mathbf{\Omega} = (\mathbb{D}, d\mathbf{s}), \qquad d\mathbf{s}(z)^2 = a(z)\,dx^2 + b(z)\,dy^2 + 2\,c(z)\,dxdy, \tag{1.1}$$

where we use the convention $z = x + iy$. The smoothness of the metric means that a, b, c are C^∞-smooth real-valued functions on the closed disk $\bar{\mathbb{D}}$, subject to the Riemannian metric conditions

$$0 < a(z), b(z), \qquad c(z)^2 < a(z)\,b(z).$$

We are interested in the problem of reconstructing the metric, if its associated curvature form is given. The *area form* is given by

$$d\mathbf{\Sigma}(z) = \big(a(z)b(z) - c(z)^2\big)\,d\Sigma(z),$$

where

$$d\Sigma(z) = \frac{dxdy}{\pi}, \qquad z = x + iy.$$

The Gaussian *curvature function* $\boldsymbol{\kappa}$ is a geometric quantity, which is given by Brioschi's formula in terms of the coordinatization as

$$\boldsymbol{\kappa} = \frac{1}{(a(z)b(z) - c(z)^2)^2}\Bigg\{ \\ \det\begin{pmatrix} -\frac{1}{2}\partial_y^2 a(z) - \frac{1}{2}\partial_x^2 b(z) + \partial_{xy}^2 c(z) & \frac{1}{2}\partial_x a(z) & \partial_x c(z) - \frac{1}{2}\partial_y a(z) \\ \partial_y c(z) - \frac{1}{2}\partial_x b(z) & a(z) & c(z) \\ \frac{1}{2}\partial_y b(z) & c(z) & b(z) \end{pmatrix} \\ - \det\begin{pmatrix} 0 & \frac{1}{2}\partial_y a(z) & \frac{1}{2}\partial_x b(z) \\ \frac{1}{2}\partial_y a(z) & a(z) & c(z) \\ \frac{1}{2}\partial_x b(z) & c(z) & b(z) \end{pmatrix} \Bigg\}$$

The *curvature form* is given by the expression

$$\mathbf{K} = \boldsymbol{\kappa}\,d\mathbf{\Sigma};$$

it measures the distribution of the curvature in space. On the curved surface $\mathbf{\Omega}$, there exists a counterpart of the usual Laplacian in the plane, known as the *Laplace-Beltrami operator*, denoted by $\mathbf{\Delta}$. In terms of the given coordinates, it can be expressed by

$$\mathbf{\Delta} = \frac{1}{4\sqrt{a(z)b(z) - c(z)^2}}\Bigg\{\partial_x\Bigg[\frac{b(z)}{\sqrt{a(z)b(z) - c(z)^2}}\,\partial_x - \frac{c(z)}{\sqrt{a(z)b(z) - c(z)^2}}\,\partial_y\Bigg] \\ + \partial_y\Bigg[-\frac{c(z)}{\sqrt{a(z)b(z) - c(z)^2}}\,\partial_x + \frac{a(z)}{\sqrt{a(z)b(z) - c(z)^2}}\,\partial_y\Bigg]\Bigg\}.$$

We say that a twice differentiable function f on the curved surface $\mathbf{\Omega}$ is *harmonic* – or Laplace-Beltrami harmonic, if we want to emphasize that we use

the Laplacian induced by the metric – provided that $\mathbf{\Delta} f(z)=0$ holds throughout $\mathbf{\Omega}$. Now, suppose we have two curved surfaces $\mathbf{\Omega}$ and $\mathbf{\Omega}'$, which are both modelled by the unit disk $\mathbb{D}$:

$$\mathbf{\Omega}=(\mathbb{D}, d\mathbf{s}), \qquad \mathbf{\Omega}'=(\mathbb{D}, d\mathbf{s}').$$

As the corresponding Laplace-Beltrami operators $\mathbf{\Delta}$ and $\mathbf{\Delta}'$ are generally different, we should expect them to give rise to different collections of harmonic functions. If they give rise to identical collections of harmonic functions, we say that the metrics $d\mathbf{s}$ and $d\mathbf{s}'$ are *isoharmic*. This relation between two metrics of been isoharmic is an equivalence relation and its equivalence classes are called *isoharmic classes of metrics*. Let $\mathfrak{M}$ denote the collection of all C^∞-smooth metrics on $\bar{\mathbb{D}}$, and let $\mathfrak{M}_\alpha$ run through all the isoharmic classes as α passes through a suitably large index set. We then get the disjoint decomposition

$$\mathfrak{M}=\bigcup_\alpha \mathfrak{M}_\alpha,$$

which provides a fibering of $\mathfrak{M}$. Of particular interest is the fiber $\mathfrak{M}_0$ (assuming that the index set contains the value 0) which contains the Euclidean metric as an element. We claim that $\mathfrak{M}_0$ coincides with the collection of so-called isothermal metrics. We recall the standard terminology that $d\mathbf{s}$ is *isothermal* if

$$d\mathbf{s}(z)^2=\omega(z)\,|dz|^2=\omega(z)\left(dx^2+dy^2\right), \tag{1.2}$$

holds for some C^∞-smooth function $\omega(z)$ which is positive at each point of $\bar{\mathbb{D}}$. This means in terms of the functions a, b, c that

$$a(z)=b(z)=\omega(z) \quad \text{and} \quad c(z)=0,$$

hold throughout $\mathbb{D}$. The area form is then $d\mathbf{\Sigma}=\omega d\Sigma$. The Laplace-Beltrami operator becomes a slight variation of the ordinary Laplacian

$$\mathbf{\Delta}=\frac{1}{\omega(z)}\,\Delta,$$

where Δ denotes the normalized Laplacian

$$\Delta=\Delta_z=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \quad z=x+\mathrm{i}y.$$

This means that the Laplace-Beltrami harmonic functions for an isothermal metric are just the ordinary harmonic functions. In other words, the isothermal metrics are contained in $\mathfrak{M}_0$. To see that all metrics in $\mathfrak{M}_0$ are isothermal, we pick a metric $d\mathbf{s}$ in $\mathfrak{M}_0$ and note that the functions $f(z;z_0)=\mathrm{Re}\left[(z-z_0)^2\right]$ and $g(z;z_0)=\mathrm{Im}\left[(z-z_0)^2\right]$ are ordinary harmonic functions of z, and therefore also Laplace-Beltrami harmonic functions. We obtain the equations $\mathbf{\Delta}_z f(z;z_0)=0$ and $\mathbf{\Delta}_z g(z;z_0)=0$ and implement the above formula for the Laplace-Beltrami

operator $\boldsymbol{\Delta}$. We evaluate the Laplace-Beltrami equation at the point z_0; as we vary the point z_0 in $\mathbb{D}$, the assertion that the metric is isothermal follows.

We would like to better understand the other isoharmic classes $\mathfrak{M}_\alpha$. To this end, we pick a metric $d\mathbf{s}_1$ in $\mathfrak{M}_\alpha$, for a fixed index α. It is well-known in the theory of quasi-conformal mappings that it is possible to find a C^∞-homeomorphism F of $\bar{\mathbb{D}}$ which changes the coordinate chart so that the metric $d\mathbf{s}_1$ becomes isothermal [1]. Under the same coordinate change F, the other elements of $\mathfrak{M}_\alpha$ change as well. However, the collection of Laplace-Beltrami harmonic functions, being determined by the geometry of the abstract surface in question, remains the same, except for the obvious composition with the chart function F. The Laplace-Beltrami harmonic functions for the metric $d\mathbf{s}_1$ are after the coordinate change just the ordinary harmonic functions, and this then carries over to all the other metrics in $\mathfrak{M}_\alpha$. In other words, the C^∞-homeomorphism F effects an identification of $\mathfrak{M}_\alpha$ with $\mathfrak{M}_0$.

The optimization problem. We are given a C^∞-smooth real-valued 2-form $\boldsymbol{\mu}$ on the closed unit disk $\bar{\mathbb{D}}$, and we are looking for a metric $d\mathbf{s}$ on $\mathbb{D}$ of the type (1.1) which is smooth up to the boundary and has $\boldsymbol{\mu}$ as curvature form $\mathbf{K} = \boldsymbol{\kappa}\, d\boldsymbol{\Sigma}$. There are plenty of such metrics. To reduce their number, we decide to minimize the *total area* of the associated surface $\boldsymbol{\Omega} = (\mathbb{D}, d\mathbf{s})$ under two additional conditions. We require that (1) the area form at the origin is essentially the area form of the plane, that is,

$$a(0)b(0) - c(0)^2 = 1,$$

and we also ask that (2) we only minimize over a fixed isoharmic class $\mathfrak{M}_\alpha$ of metrics. This second requirement means that we fix the collection of harmonic functions in the unit disk while performing the minimization. A natural question here is whether there exists a minimizing surface, and whether it has a smooth boundary.

In order to solve this problem analytically, it is very helpful to note that by using the C^∞-homeomorphism F, mentioned in the previous subsection, we may choose to work only with the class $\mathfrak{M}_0$ of isothermal metrics. This means that the metrics are of the form

$$d\mathbf{s}(z)^2 = \omega(z)\,|dz|^2 = \omega(z)\left(dx^2 + dy^2\right),$$

for some C^∞-smooth function $\omega(z)$ which is positive at each point of $\bar{\mathbb{D}}$. The formula for the curvature simplifies greatly,

$$\boldsymbol{\kappa}(z) = -2\boldsymbol{\Delta}\log\omega(z) = -\frac{2}{\omega(z)}\Delta\log\omega(z), \qquad z \in \mathbb{D},$$

and so does the formula for the curvature form:

$$\mathbf{K}(z) = -2\boldsymbol{\Delta}\log\omega(z)\, d\boldsymbol{\Sigma}(z) = -2\,\Delta\log\omega(z)\, d\Sigma(z), \qquad z \in \mathbb{D}. \tag{1.3}$$

Let us now see how this coordinatization simplifies the formulation of our problem. We are given a C^∞-smooth real-valued function μ on the closed unit disk, which is the density function for the 2-form $\boldsymbol{\mu}$:

$$\boldsymbol{\mu}(z) = \mu(z)\, d\Sigma(z), \qquad z \in \mathbb{D}. \tag{1.4}$$

We write the curvature form $\mathbf{K}$ as

$$\mathbf{K}(z) = K(z)\, d\Sigma(z), \qquad K(z) = -2\,\Delta \log \omega(z).$$

The equation $\mathbf{K} = \boldsymbol{\mu}$ then becomes

$$-2\,\Delta \log \omega(z) = \mu(z), \qquad z \in \mathbb{D}.$$

As we have already fixed the isoharmic class by setting the Laplace-Beltrami harmonic functions equal to the ordinary harmonic functions in the plane, all that we really need to fix is the value of ω at the origin, while minimizing the total area of the surface $\mathbf{\Omega} = (\mathbb{D}, d\mathbf{s})$. However, *it is equivalent to maximize the value $\omega(0)$ while keeping the total area constant*, and we find it convenient to fix the latter to equal 1:

$$|\mathbf{\Omega}|_{\mathbf{\Sigma}} = |\mathbb{D}|_{\mathbf{\Sigma}} = \int_{\mathbb{D}} d\mathbf{\Sigma}(z) = \int_{\mathbb{D}} \omega(z)\, d\Sigma(z) = 1.$$

The optimization problem and the mean value property. We recall that the weight ω should solve the following problem:

$$\begin{cases} \text{maximize} \quad \omega(0), \quad \text{while} \\ \Delta \log \omega(z) = -\dfrac{1}{2}\,\mu(z), \qquad z \in \mathbb{D}, \quad \text{and} \\ \displaystyle\int_{\mathbb{D}} \omega(z)\, d\Sigma(z) = 1. \end{cases} \tag{OP}$$

Here, μ is a C^∞-smooth real-valued function on $\bar{\mathbb{D}}$, and we ask of ω that it too should be C^∞-smooth (and positive) on $\bar{\mathbb{D}}$. It is not clear that the optimization problem (OP) should have such a nice solution in general. As a matter of fact, it is possible to construct counterexamples that are fairly elementary. Nevertheless, we shall investigate the properties that such an extremal weight $\omega = \omega_0$ should enjoy. We compare the extremal weight ω_0 with nearby weights ω_t, of the form

$$\omega_t(z) = e^{th(z)}\, \omega_0(z),$$

where h is a C^∞-smooth real-valued function on $\bar{\mathbb{D}}$ that is harmonic in the interior $\mathbb{D}$, with $h(0) = 0$, and t is a real parameter. Then

$$\Delta \log \omega_t(z) = \Delta \log \omega_0(z) = -\frac{1}{2}\,\mu(z), \qquad z \in \mathbb{D},$$

and $\omega_t(0)=\omega_0(0)$. The extremal property of ω_0 now forces the inequality

$$\int_{\mathbb{D}} \omega_0(z)\, d\Sigma(z) \le \int_{\mathbb{D}} \omega_t(z)\, d\Sigma(z) = \int_{\mathbb{D}} e^{th(z)}\, \omega_0(z)\, d\Sigma(z) \tag{1.5}$$

to hold. By Taylor's formula,

$$e^{th(z)} = 1 + th(z) + O(t^2),$$

for t close to 0. As we plug this into equation (1.5), we arrive at

$$\int_{\mathbb{D}} \omega_0(z)\, d\Sigma(z) \le \int_{\mathbb{D}} \omega_0(z)\, d\Sigma(z) + t \int_{\mathbb{D}} h(z)\, \omega_0(z)\, d\Sigma(z) + O(t^2). \tag{1.6}$$

By varying t from small positive to small negative values, we realize that the only way for (1.6) to hold is if

$$\int_{\mathbb{D}} h(z)\, \omega_0(z)\, d\Sigma(z) = 0.$$

If we drop the requirement on h that $h(0)=0$, and consider the function $h(z)-h(0)$ instead in the above argument, we obtain

$$\int_{\mathbb{D}} h(z)\, \omega_0(z)\, d\Sigma(z) = h(0).$$

This is what we call the *mean value property* of the weight ω_0. Note that by an approximation argument, the above mean value property remains valid when we extend the collection of h to all harmonic functions in $\mathbb{D}$ that are integrable with respect to area measure. We find that we are looking for a (positive) weight ω_0 that is C^∞-smooth up to the boundary, with

$$\Delta \log \omega_0(z) = -\frac{1}{2}\, \mu(z), \qquad z \in \mathbb{D}, \tag{1.7}$$

and the mean value property

$$\int_{\mathbb{D}} h(z)\, \omega_0(z)\, d\Sigma(z) = h(0), \qquad h \in \mathcal{H}^1(\mathbb{D}), \tag{1.8}$$

where $\mathcal{H}^1(\mathbb{D})$ stands for the Banach space of all complex-valued area integrable harmonic functions on $\mathbb{D}$. Strictly speaking, there is more information contained in the extremal property of ω_0, but clearly, if we find a unique C^∞-smooth solution ω_0 to (1.7) and (1.8), then it is definitely our prime candidate for the solution to the optimization problem (OP).

The relationship with Bergman kernel functions. We now discuss how to actually find the extremal weight ω_0; more precisely, we discuss the problem of solving the equations (1.7) and (1.8). First, we note that by elementary potential theory, one solution ω_1 to (1.7) is given by

$$\log \omega_1(z) = -\int_{\mathbb{D}} \log \left| \frac{z-w}{1-z\bar{w}} \right| \mu(w)\, d\Sigma(w), \qquad z \in \mathbb{D}, \tag{1.9}$$

and it is well known that $\log \omega_1$ is real-valued and C^∞-smooth on $\bar{\mathbb{D}}$, because μ has these properties. Any solution to (1.7), then, has the form

$$\log \omega_0(z) = \log \omega_1(z) + H(z), \qquad z \in \mathbb{D},$$

where H is real-valued and harmonic in $\mathbb{D}$; given the smoothness assumptions on ω_0, H should be C^∞-smooth on $\bar{\mathbb{D}}$. We find a holomorphic function F on $\mathbb{D}$, which is zero-free and C^∞-smooth up to the boundary, such that

$$\log |F(z)|^2 = H(z), \qquad z \in \mathbb{D}.$$

By restricting (1.8) to holomorphic functions, we obtain

$$\int_D f(z)\, |F(z)|^2 \omega_1(z)\, d\Sigma(z) = f(0), \tag{1.10}$$

for all f in $\mathcal{A}^1(\mathbb{D})$, the space of area-integrable holomorphic functions on $\mathbb{D}$. Let $g \in \mathcal{A}^1(\mathbb{D})$ be arbitrary, except that $g(0) = 0$; then $f = g/F$ is in $\mathcal{A}^1(\mathbb{D})$ as well, and we find that (1.10) states that

$$\int_D g(z)\, \bar{F}(z)\, \omega_1(z)\, d\Sigma(z) = 0. \tag{1.11}$$

We interpret this in terms of the Hilbert space $\mathcal{A}^2(\mathbb{D}, \omega_1)$, consisting of the square area-integrable holomorphic functions on $\mathbb{D}$, supplied with the weighted norm

$$\|f\|_{\omega_1} = \left\{ \int_{\mathbb{D}} |f(z)|^2\, \omega_1(z)\, d\Sigma(z) \right\}^{1/2}.$$

This space $\mathcal{A}^2(\mathbb{D}, \omega_1)$ is known as a *weighted Bergman space*. Equation (1.11) then states that F is perpendicular to all the functions

$$\big\{ g \in \mathcal{A}^2(\mathbb{D}, \omega_1) :\, g(0) = 0 \big\},$$

and this means that F is of the form

$$F(z) = C\, K_{\omega_1}(z, 0),$$

where C is a complex constant, and $K_{\omega_1}(z, w)$ is the *weighted Bergman kernel with weight* ω_1. We recall that the weighted Bergman kernel is defined by

$$K_{\omega_1}(z, w) = \sum_{n=1}^{+\infty} e_n(z)\, \bar{e}_n(w), \qquad z, w \in \mathbb{D},$$

where the functions $e_1(z)$, $e_2(z)$, $e_3(z), \ldots$ run through an orthonormal basis for $\mathcal{A}^2(\mathbb{D}, \omega_1)$ [4, pp. 43-44]. Alternatively, $K_{\omega_1}(\cdot, w)$ is the unique element of $\mathcal{A}^2(\mathbb{D}, \omega_1)$ which supplies the point evaluation at $w \in \mathbb{D}$:

$$f(w) = \int_{\mathbb{D}} f(z)\, \bar{K}_{\omega_1}(z, w)\, \omega_1(z)\, d\Sigma(z),$$

for $f \in \mathcal{A}^2(\mathbb{D}, \omega_1)$ [3]. The constant C is easily determined (at least in modulus) by applying (1.10) with the choice $f = 1$, which leads to

$$F(z) = K_{\omega_1}(0,0)^{-1/2}\, K_{\omega_1}(z,0), \qquad z \in \mathbb{D}. \tag{1.12}$$

Returning back to the extremal weight ω_0, we find that it is of the form

$$\omega_0(z) = \frac{\left|K_{\omega_1}(z,0)\right|^2}{K_{\omega_1}(0,0)}\, \omega_1(z), \qquad z \in \mathbb{D}. \tag{1.13}$$

By the elliptic regularity theory for PDEs [14], the function $K_{\omega_1}(\cdot, 0)$ is C^∞-smooth on $\bar{\mathbb{D}}$. We realize, then, that *a necessary condition for the existence of a C^∞-smooth positive weight ω_0 on $\bar{\mathbb{D}}$ that solves our problem (OP) is that*

$$K_{\omega_1}(z,0) \neq 0, \qquad z \in \bar{\mathbb{D}}. \tag{1.14}$$

It turns out that it is also sufficient, and that the extremal solution is then given by (1.13). For, if ω is another weight that solves (1.7), and is C^∞-smooth and positive on $\bar{\mathbb{D}}$, then it is of the form

$$\omega(z) = |F(z)|^2\, \omega_1(z), \qquad z \in \mathbb{D},$$

where F is C^∞-smooth on $\bar{\mathbb{D}}$, analytic in $\mathbb{D}$, and zero-free in $\bar{\mathbb{D}}$. It is well-known that the function

$$F(z) = K_{\omega_1}(0,0)^{-1/2}\, K_{\omega_1}(z,0), \qquad z \in \mathbb{D},$$

is the unique (up to multiplication by unimodular constants) solution to the extremal problem to maximize $|F(0)|$, given that F is holomorphic in $\mathbb{D}$ and

$$\int_{\mathbb{D}} |F(z)|^2\, \omega_1(z)\, d\Sigma(z) = 1,$$

which means that $\omega(0) \leq \omega_0(0)$ for all competitors ω with

$$\int_{\mathbb{D}} \omega(z)\, d\Sigma(z) = 1.$$

This shows that ω_0, as given by (1.13), is indeed the solution to the extremal problem, provided that condition (1.14) is fulfilled.

A toy example. We should study a simple example, to develop some intuition. We consider the degenerate data

$$\mu(z) = -\theta\, \delta_\lambda(z),$$

where $\delta_\lambda(z)$ stands for the Dirac delta function concentrated at the point $\lambda \in \mathbb{D}$, and θ is a real parameter. This does not fulfill our smoothness requirement at

the point λ, so we should think of it as a limit case of smooth functions μ_n. The weight function ω_1 supplied by equation (1.9) is explicitly given by

$$\omega_1(z) = \left|\frac{z-\lambda}{1-\bar{\lambda}z}\right|^\theta, \qquad z \in \mathbb{D},$$

which is a reasonable weight on $\mathbb{D}$ provided that $-2 < \theta < +\infty$; outside this interval, the weight fails to be area-summable near the point λ. We need to find the weighted Bergman kernel function K_{ω_1}. To this end, we note first that if ϕ is a Mœbius automorphism of $\mathbb{D}$, then we have the following relationship between the kernel functions for the weights ω and $\omega \circ \phi$:

$$K_{\omega\circ\phi}(z,w) = \phi'(z)\,\bar{\phi}'(w)\,K_\omega\big(\phi(z),\phi(w)\big), \qquad z,w \in \mathbb{D}.$$

Let ω_2 stand for the radial weight

$$\omega_2(z) = |z|^\theta, \qquad z \in \mathbb{D};$$

then the weighted Bergman space $\mathcal{A}^2(\mathbb{D},\omega_2)$ has the reproducing kernel function

$$K_{\omega_2}(z,w) = \frac{1}{(1-z\bar{w})^2} + \frac{\theta}{2}\,\frac{1}{1-z\bar{w}}, \qquad z,w \in \mathbb{D}.$$

As $\omega_1 = \omega_2 \circ \phi$, where ϕ is the involutive Mœbius automorphism

$$\phi(z) = \frac{\lambda - z}{1-\bar{\lambda}z}, \qquad z \in \mathbb{D},$$

we find that

$$K_{\omega_1}(z,w) = \frac{1}{(1-z\bar{w})^2} + \frac{\theta}{2}\,\frac{1-|\lambda|^2}{(1-\bar{\lambda}z)(1-\lambda\bar{w})(1-z\bar{w})}, \qquad z,w \in \mathbb{D}.$$

Plugging in $w=0$, we obtain

$$K_{\omega_1}(z,0) = 1 + \frac{\theta}{2}\,\frac{1-|\lambda|^2}{1-\bar{\lambda}z}, \qquad z \in \mathbb{D}.$$

This expression has a zero in $\mathbb{D}$ precisely when (recall that $-2 < \theta < +\infty$ is assumed)

$$-2 < \theta < -\frac{2}{1+|\lambda|}.$$

In particular, if $-2 < \theta < -1$, we may choose λ close to the unit circle $\mathbb{T}$, to make sure that $K_{\omega_1}(z,0)$ has a zero in $\mathbb{D}$, whereas if $-1 \le \theta < +\infty$, no such zero can be found, no matter how cleverly we try to pick $\lambda \in \mathbb{D}$. This means that the function ω_0 supplied by relation (1.13) is C^∞-smooth on $\bar{\mathbb{D}}\setminus\{\lambda\}$ for $-1 \le \theta < +\infty$, which constitutes the solution to our extremal problem. However, for $-2 < \theta < -1$, we see that it is possible to pick $\lambda \in \mathbb{D}$ so that no smooth solution to the extremal problem will exist.

This calculation suggests a pattern: *for hyperbolic metrics (this means that the Gaussian curvature is negative everywhere), we have a smooth solution ω_0 to the extremal problem, whereas when the metric becomes elliptic (positive Gaussian curvature), we tend to get in trouble.*

Statement of the main results. We are going to compare the curvature form with that of the hyperbolic plane, and use this as a criterion for how strongly curved our metric is. The Poincaré metric is

$$d\mathbf{s}_{\mathbb{H}}(z) = \frac{2\,|dz|}{1-|z|^2}, \qquad z \in \mathbb{D},$$

and the associated area form is

$$d\boldsymbol{\Sigma}_{\mathbb{H}}(z) = \frac{4\,d\Sigma(z)}{(1-|z|^2)^2}.$$

The curvature form for the Poincaré metric (which supplies the standard model for the hyperbolic plane $\mathbb{H}$) is

$$\mathbf{K}_{\mathbb{H}}(z) = -\frac{4\,d\Sigma(z)}{(1-|z|^2)^2};$$

we are to compare the curvature form data $\boldsymbol{\mu}(z) = \mu(z)\,d\Sigma(z)$ with $\mathbf{K}_{\mathbb{H}}(z)$.

THEOREM 1.1. *Let μ be a C^∞-smooth real-valued function on $\bar{\mathbb{D}}$, and suppose that the associated 2-form $\boldsymbol{\mu}(z) = \mu(z)\,d\Sigma(z)$ has*

$$\boldsymbol{\mu}(z) + \frac{1}{2}\,\mathbf{K}_{\mathbb{H}}(z) \le 0, \qquad z \in \mathbb{D}.$$

Then the optimization problem (OP) has a unique C^∞-smooth positive solution ω_0 on $\bar{\mathbb{D}}$, which is given by relation (1.13), where ω_1 is as in (1.9). In addition,

$$K_{\omega_1}(z,w) \neq 0, \qquad (z,w) \in \big(\bar{\mathbb{D}} \times \mathbb{D}\big) \cup \big(\mathbb{D} \times \bar{\mathbb{D}}\big).$$

This is in line with the intuition we arrived at from our "toy example". Note that the assumption of the theorem is considerably weaker than requiring negative data $\boldsymbol{\mu}$. As for the necessity of the condition of Theorem 1.1, we have obtained the following.

THEOREM 1.2. *Fix α, $\alpha_0 < \alpha < +\infty$, where $\alpha_0 \approx 1.04$. Consider a C^∞-smooth real-valued function μ on $\bar{\mathbb{D}}$, and suppose that the associated 2-form $\boldsymbol{\mu}(z) = \mu(z)\,d\Sigma(z)$ has*

$$\boldsymbol{\mu}(z) + \frac{\alpha}{2}\,\mathbf{K}_{\mathbb{H}}(z) \le 0, \qquad z \in \mathbb{D}.$$

Then there exists a choice of μ such that the optimization problem (OP) fails to have a C^∞-smooth positive solution ω_0 on $\bar{\mathbb{D}}$.

We conjecture that Theorem 1.2 will remain true with $\alpha_0 = 1$, making the statement of Theorem 1.1 essentially sharp.

Work related to the problems considered here can be found in the papers [6], [7], [9], [5], [8], [15], [16], [11], [17].

The metric potential. The Green function for the Laplacian Δ is

$$G(z,w) = \log \left| \frac{z-w}{1-z\bar{w}} \right|^2, \qquad z,w \in \mathbb{D}, \quad z \neq w.$$

The *metric potential* associated with the isothermal metric (1.2) is the function

$$\mathbf{\Phi}(z) = \int_{\mathbb{D}} G(z,w)\,\omega(w)\,d\Sigma(w), \qquad z \in \mathbb{D};$$

see, for instance, [13]. It solves the boundary value problem ($\mathbb{T}$ is the unit circle)

$$\begin{cases} \mathbf{\Delta\Phi}(z) = 1, & z \in \mathbb{D}, \\ \mathbf{\Phi}(z) = 0, & z \in \mathbb{T}. \end{cases} \tag{1.15}$$

It is set in boldface because it expresses a quantity that is independent of the choice of coordinates. We wish to describe the potential equation (1.7) and the mean value property (1.8), which the smooth solution to the optimization problem (OP) should satisfy, in terms of the metric potential. If we let $\mathbf{\Phi}_0$ stand for the metric potential associated with a weight ω_0 with (1.7) and (1.8), we find that $\mathbf{\Phi}_0$ solves

$$\begin{cases} \Delta \log \Delta \mathbf{\Phi}_0(z) = -\dfrac{1}{2}\,\mu(z), & z \in \mathbb{D}, \\ \mathbf{\Phi}_0(z) = 0, & z \in \mathbb{T}, \\ \dfrac{\partial}{\partial n(z)}\,\mathbf{\Phi}_0(z) = 2, & z \in \mathbb{T}, \end{cases} \tag{1.16}$$

where $\partial/\partial n(z)$ denotes the normal derivative, taken in the exterior direction. We are of course only looking for subharmonic solutions $\mathbf{\Phi}_0$, which means that the expression $\log \Delta \mathbf{\Phi}_0$ is more or less well-defined (at least if we have some additional smoothness, and the subharmonicity is "strong"). The normal derivative condition in (1.16) cleverly encodes the mean value property (1.8), as is seen easily from an application of Green's formula. *We realize that Theorem 1.1 can be interpreted as an assertion claiming the existence of a unique smooth solution to the non-linear elliptic boundary value problem (1.16) under appropriate conditions on* μ. By moving the point at the origin around by applying a Mœbius transformation that fixes the unit disk, we find that the method can also treat the case when the data for the normal derivative is replaced by

$$\frac{\partial}{\partial n(z)}\,\mathbf{\Phi}_0(z) = 2\,\frac{1-|\lambda|^2}{|\lambda - z|^2},$$

where λ is any point of $\mathbb{D}$. It would be interesting to have an analysis of the equation (1.16), where the boundary data are considered from a wider class of functions.

2 Preliminaries

The standard weighted Bergman spaces. For $-1 < \alpha < +\infty$, let $\mathcal{A}^2_\alpha(\mathbb{D}) = \mathcal{A}^2(\mathbb{D}, \omega_\alpha)$ denote the weighted Bergman space for the weight

$$\omega_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha, \qquad z \in \mathbb{D}.$$

The norm in $\mathcal{A}^2_\alpha(\mathbb{D})$ is written

$$\|f\|_\alpha = \left\{ \int_{\mathbb{D}} |f(z)|^2\, \omega_\alpha(z)\, d\Sigma(z) \right\}^{1/2}.$$

Take a point $\lambda \in \mathbb{D} \setminus \{0\}$, and let $\varphi_\lambda(z) = \varphi_{\lambda,\alpha}(z)$ be the so-called *extremal function* for the problem

$$\sup \Big\{ \operatorname{Re} f(0) :\ f(\lambda) = 0,\ \|f\|_\alpha \le 1 \Big\},$$

which is unique, by elementary Hilbert space theory. If we let $K_\lambda = K^\alpha_\lambda$ denote the reproducing kernel for the closed subspace

$$\Big\{ f \in \mathcal{A}^2_\alpha(\mathbb{D}) :\ f(\lambda) = 0 \Big\},$$

then the extremal function can be written

$$\varphi_\lambda(z) = K_\lambda(0,0)^{-1/2}\, K_\lambda(z,0), \qquad z \in \mathbb{D}.$$

When the function φ_λ serves as a good divisor of the zero at λ in the space $\mathcal{A}^2_\alpha(\mathbb{D})$, it is called the *canonical divisor* of λ as it is called in [6], or the *contractive zero divisor*, provided that division by it defines a norm contractive operation as it is called in [7]. It is known [10, pp. 58] (see Section 4 for details) that the reproducing kernel $K = K_\alpha$ for the space $\mathcal{A}^2_\alpha(\mathbb{D})$ has the form

$$K(z,w) = \frac{1}{(1-z\bar{w})^{\alpha+2}}, \qquad z, w \in \mathbb{D},$$

and that the kernel K_λ is derived from K via the identity [10]

$$\begin{aligned} K_\lambda(z,w) = K(z,w) - \frac{K(z,\lambda)K(\lambda,w)}{K(\lambda,\lambda)} \\ = \frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{(1-|\lambda|^2)^{\alpha+2}}{(1-z\bar{\lambda})^{\alpha+2}(1-\lambda\bar{w})^{\alpha+2}}, \end{aligned}$$

(see Section 4 for the details). It follows that the extremal function is given by

$$\varphi_\lambda(z) = \big(1-(1-|\lambda|^2)^{\alpha+2}\big)^{-1/2} \left\{ 1 - \frac{(1-|\lambda|^2)^{\alpha+2}}{(1-z\bar{\lambda})^{\alpha+2}} \right\}, \qquad z \in \mathbb{D}. \tag{2.1}$$

The following property of the extremal function $\varphi_\lambda = \varphi_{\lambda,\alpha}$ is fundamental.

LEMMA 2.1. *Fix α in the interval $-1 < \alpha < +\infty$. Then, for each bounded harmonic function h on $\mathbb{D}$, we have that*

$$\int_{\mathbb{D}} h(z)\, |\varphi_\lambda(z)|^2 \, (\alpha + 1) \left(1 - |z|^2\right)^\alpha d\Sigma(z) = h(0).$$

Proof. First, suppose h is an analytic polynomial. Then, if we recall the definition of φ_λ in terms of K_λ, and use the reproducing property of the kernel K_λ, we have that

$$\begin{aligned} &\int_{\mathbb{D}} h(z)\, |\varphi_\lambda(z)|^2 (\alpha + 1)\left(1 - |z|^2\right)^\alpha d\Sigma(z) \\ &\qquad = \int_{\mathbb{D}} h(z) \frac{K_\lambda(z,0)}{K_\lambda(0,0)} \bar{K}_\lambda(z,0)\, (\alpha+1)\left(1 - |z|^2\right)^\alpha d\Sigma(z) = h(0). \end{aligned} \tag{2.2}$$

Taking complex conjugates in (2.2), we realize that the desired equality holds for all harmonic polynomials (defined to be sums of analytic and antianalytic polynomials). A simple approximation argument finishes the proof. □

A weighted biharmonic Green function. The biharmonic Green function is the function Γ on $\mathbb{D} \times \mathbb{D}$ that solves the boundary value problem

$$\begin{cases} \Delta_z^2 \Gamma(z,w) = \delta_w(z), \qquad z \in \mathbb{D}, \\ \Gamma(z,w) = 0, \qquad z \in \mathbb{T}, \\ \dfrac{\partial}{\partial n(z)} \Gamma(z,w) = 0, \qquad z \in \mathbb{T}, \end{cases} \tag{2.3}$$

where $w \in \mathbb{D}$, and δ_w stands for the unit point mass at the point w. We will think of locally summable functions f on some domain Ω of the complex plane as distributions on Ω via the linear duality

$$\langle f, \phi \rangle = \int_\Omega f(z)\, \phi(z)\, d\Sigma(z),$$

for compactly supported test functions ϕ on Ω. Given this normalization, the biharmonic Green function is given explicitly by the formula

$$\Gamma(z,w) = |z-w|^2 \log \left| \frac{z-w}{1 - z\bar{w}} \right|^2 + (1 - |z|^2)(1 - |w|^2), \qquad z, w \in \mathbb{D}.$$

We shall also need the Green function for the weighted biharmonic operator $\Delta(1 - |z|^2)^{-1} \Delta$, denoted Γ_1, which, by definition, solves, for fixed $w \in \mathbb{D}$,

$$\begin{cases} \Delta(1 - |z|^2)^{-1} \Delta \Gamma_1(z,w) = \delta_w(z), \qquad z \in \mathbb{D}, \\ \Gamma_1(z,w) = 0, \qquad z \in \mathbb{T}, \\ \dfrac{\partial}{\partial n(z)} \Gamma_1(z,w) = 0, \qquad z \in \mathbb{T}, \end{cases} \tag{2.4}$$

Although the differential operator is singular at the boundary, the above boundary value problem has a unique solution (we may use Green's theorem to interpret the boundary data in terms of integral conditions for $\Delta_z\Gamma_1(z,w)$, which are uniquely solvable).

The function Γ_1 was calculated explicitly in [9]:

LEMMA 2.2. *We have that*

$$\Gamma_1(z,w) = \left\{|z-w|^2 - \frac{1}{4}\left|z^2-w^2\right|^2\right\}\log\left|\frac{z-w}{1-z\bar{w}}\right|^2 + \frac{1}{8}\,(1-|z|^2)(1-|w|^2)\left\{7-|z|^2-|w|^2-|zw|^2-4\mathrm{Re}\,z\bar{w} - 2\,(1-|z|^2)(1-|w|^2)\frac{1-|zw|^2}{|1-z\bar{w}|^2}\right\},$$

for $z,w\in\mathbb{D}$. Moreover, it follows that, for $z,w\in\mathbb{D}$,

$$\frac{1}{8}\,\frac{(1-|z|^2)^3(1-|w|^2)^3}{|1-z\bar{w}|^2} \le \Gamma_1(z,w) \le \frac{1}{8}\,\frac{(1-|z|^2)^3(1-|w|^2)^3}{|1-z\bar{w}|^4}\left\{|1-z\bar{w}|^2+4-|z+w|^2\right\}.$$

Proof. Let $\widetilde{\Gamma}_1$ stand for the function defined by the above expression; we want to show that $\widetilde{\Gamma}_1=\Gamma_1$. To this end, we show that it solves the boundary value problem which determines Γ_1. We first note that

$$\Delta_z\widetilde{\Gamma}_1(z,w) = (1-|z|^2)\big[G(z,w)+H_1(z,w)\big], \qquad z,w\in\mathbb{D},$$

where $H_1(z,w)$ is the harmonic function of z which is given by

$$H_1(z,w) = (1-|w|^2)\left\{\frac{1}{2}\,(3-|w|^2)\,\frac{1-|zw|^2}{|1-z\bar{w}|^2} + (1-|w|^2)\,\mathrm{Re}\,\left[\frac{z\bar{w}}{(1-z\bar{w})^2}\right]\right\},$$

for $z,w\in\mathbb{D}$. Thus, $\widetilde{\Gamma}_1$ satisfies

$$\Delta(1-|z|^2)^{-1}\Delta\widetilde{\Gamma}_1(z,w) = \delta_w(z), \qquad z\in\mathbb{D}.$$

We shall now establish the specified inequality for $\widetilde{\Gamma}_1$, which entails that

$$\widetilde{\Gamma}_1(z,w) = O\big((1-|z|)^3\big), \qquad |z|\to 1^-.$$

It follows that for a fixed $w\in\mathbb{D}$, the function $\widetilde{\Gamma}_1$ satisfies the boundary conditions

$$\begin{cases} \widetilde{\Gamma}_1(z,w)=0, & z\in\mathbb{T},\\ \partial_{n(z)}\widetilde{\Gamma}_1(z,w)=0, & z\in\mathbb{T}, \end{cases}$$

where $\partial_{n(z)}$ is the outer normal derivative. In view of this, we conclude that $\widetilde{\Gamma}_1 = \Gamma_1$. It remains to establish the claimed bounds for $\widetilde{\Gamma}_1$, from above and from below. We use the following estimate of the logarithm:

$$\frac{r}{2} - \frac{1}{2r} < \log r < -\frac{3}{2} + 2r - \frac{r^2}{2}, \qquad 0 < r < 1,$$

with the choice

$$r = \left| \frac{z-w}{1-z\bar{w}} \right|^2,$$

and base our calculations on the identity

$$1 - \left| \frac{z-w}{1-z\bar{w}} \right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\bar{w}|^2}.$$

The steps are rather lengthy but quite elementary, and are therefore left to the reader as an exercise. □

The next result is a consequence of the positivity of the Green function Γ_1.

LEMMA 2.3. *Let u be a C^2-smooth subharmonic function on $\mathbb{D}$ such that for some real β, $0 < \beta < 2$,*

$$|u(z)| = O\left(\frac{1}{(1-|z|)^\beta}\right) \qquad as \quad |z| \to 1^-.$$

Also, let $\varphi_\lambda = \varphi_{\lambda,1}$ denote the extremal function for the point $\lambda \in \mathbb{D}$ in the space $\mathcal{A}^2_1(\mathbb{D})$. Then we have the following inequality:

$$\int_{\mathbb{D}} u(z)(1-|z|^2)\, d\Sigma(z) \le \int_{\mathbb{D}} |\varphi_\lambda(z)|^2 u(z)(1-|z|^2)\, d\Sigma(z).$$

Proof. We consider the potential function

$$\Phi_\lambda(z) = \int_{\mathbb{D}} G(z,w) \big(|\varphi_\lambda(w)|^2 - 1\big)(1-|w|^2)\, d\Sigma(w), \qquad z \in \mathbb{D},$$

which solves the problem

$$\begin{cases} \Delta\Phi_\lambda(z) = \big(|\varphi_\lambda(z)|^2 - 1\big)(1-|z|^2), \quad z \in \mathbb{D}, \\ \Phi_\lambda(z) = 0, \quad z \in \mathbb{T}. \end{cases}$$

Note that by using Green's formula as in [7], we see that the property that φ_λ has according to Lemma 2.1 may be rephrased in terms of the potential function Φ_λ:

$$\frac{\partial}{\partial n(z)} \Phi_\lambda(z) = 0, \qquad z \in \mathbb{T}.$$

It follows that Φ_λ solves the over-determined boundary value problem

$$\begin{cases} \Delta\Phi_\lambda(z) = \big(|\varphi_\lambda(z)|^2 - 1\big)(1-|z|^2), & z \in \mathbb{D}, \\ \Phi_\lambda(z) = 0, & z \in \mathbb{T}, \\ \dfrac{\partial}{\partial n(z)}\Phi_\lambda(z) = 0, & z \in \mathbb{T}. \end{cases}$$

We may remove this over-determination by increasing the degree of the elliptic operator:

$$\begin{cases} \Delta\dfrac{1}{1-|z|^2}\Delta\Phi_\lambda(z) = |\varphi_\lambda'(z)|^2, & z \in \mathbb{D}, \\ \Phi_\lambda(z) = 0, & z \in \mathbb{T}, \\ \dfrac{\partial}{\partial n(z)}\Phi_\lambda(z) = 0, & z \in \mathbb{T}. \end{cases}$$

This problem has a unique solution, which may be expressed in terms of the Green function Γ_1,

$$\Phi_\lambda(z) = \int_{\mathbb{D}} \Gamma_1(z,w)\, |\varphi_\lambda'(w)|^2 d\Sigma(w) \geq 0, \quad z \in \mathbb{D}.$$

We first apply this to the case when u is C^2-smooth on $\bar{\mathbb{D}}$, and see that in view of the assumption that u is subharmonic, we find, by an application of Green's theorem, that

$$\int_{\mathbb{D}} \big(|\varphi_\lambda(z)|^2 - 1\big)\, u(z)\, (1-|z|^2)\, d\Sigma(z) = \int_{\mathbb{D}} \Phi_\lambda(z)\, \Delta u(z)\, d\Sigma(z) \geq 0.$$

If u is not smooth up to the boundary, we perform the above for the dilated function $u_r(z) = u(rz)$, with $0 < r < 1$. By the growth assumption on u and the smoothness of the function φ up to the boundary, we may let $r \to 1^-$ and apply Lebesgue's dominated convergence theorem, to conclude that

$$\int_{\mathbb{D}} \big(|\varphi_\lambda(z)|^2 - 1\big)\, u(z)\, (1-|z|^2)\, d\Sigma(z) \geq 0.$$

The proof is complete. $\square$

Now, let ω be a strictly positive C^∞-smooth weight on $\bar{\mathbb{D}}$, which is such that the function

$$z \mapsto \log\frac{\omega(z)}{1-|z|^2}$$

is subharmonic on $\mathbb{D}$. As before, let $K_\omega(z,w)$ denote the reproducing kernel for the weighted Bergman space $\mathcal{A}^2(\mathbb{D},\omega)$. The following result is basic for our further considerations.

THEOREM 2.4. *The function K_ω extends to be C^∞-smooth on the set $(\bar{\mathbb{D}} \times \mathbb{D}) \cup (\mathbb{D} \times \bar{\mathbb{D}})$.*

This is a classical theorem of elliptic regularity type, obtained in 1955 by L. Nirenberg [14]. It is independent of the above subharmonicity requirement.

Next, we consider the function $\Lambda_\omega : \mathbb{D} \to \mathbb{R}$, as given by

$$\Lambda_\omega(z) = \frac{|K_\omega(z,0)|^2}{K_\omega(0,0)} \frac{\omega(z)}{2\,(1-|z|^2)}, \qquad z \in \mathbb{D};$$

it is positive and subharmonic in $\mathbb{D}$, due to the assumptions on ω, and it has the growth behavior

$$\Lambda_\omega(z) = O\left(\frac{1}{1-|z|}\right), \qquad |z| \to 1^-.$$

A basic property of Λ_ω is the following.

LEMMA 2.5. *For each bounded harmonic function h on $\mathbb{D}$, we have that*

$$\int_{\mathbb{D}} h(z)\,\Lambda_\omega(z)\,2\big(1-|z|^2\big)\,d\Sigma(z) = h(0).$$

Proof. First, let us assume that h is an analytic polynomial. Then, using the reproducing property of the kernel function K_ω, we have that

$$\int_{\mathbb{D}} h(z)\Lambda_\omega(z)\,2\,(1-|z|^2)\,d\Sigma(z) \\ = \int_{\mathbb{D}} h(z)\frac{K_\omega(z,0)}{K_\omega(0,0)}\overline{K}_\omega(z,0)\omega(z)\,d\Sigma(z) = h(0). \quad (2.5)$$

Taking complex conjugates in (2.5), we see that the desired equality holds for all harmonic polynomials. An approximation argument finishes the proof. □

This means that the function Λ_ω has a lot in common with the function $|\varphi_\lambda|^2$ for the parameter $\alpha = 1$, which suggests it may have an expansive multiplier property that is similar to the one obtained in Lemma 2.3.

LEMMA 2.6. *Suppose that u is a C^2-smooth subharmonic function in $\mathbb{D}$, which has the growth bound*

$$|u(z)| = O\left(\frac{1}{(1-|z|)^\beta}\right) \qquad \text{as} \quad |z| \to 1^-.$$

for some real β, $0 < \beta < 1$. We then have

$$\int_{\mathbb{D}} u(z)\,2\,(1-|z|^2)\,d\Sigma(z) \le \int_{\mathbb{D}} \Lambda_\omega(z)u(z)\,2\,(1-|z|^2)\,d\Sigma(z).$$

Proof. We introduce the potential function

$$\Phi_\omega(z) = \int_{\mathbb{D}} G(z,w)\,\big[\Lambda_\omega(w) - 1\big]\,2\,(1-|w|^2)\,d\Sigma(w) \\ = \int_{\mathbb{D}} G(z,w)\left(\frac{|K_\omega(w,0)|^2}{K_\omega(0,0)}\,\omega(w) - 2\,(1-|w|^2)\right)d\Sigma(w), \qquad z \in \mathbb{D},$$

which solves the boundary value problem

$$\begin{cases} \Delta\Phi_\omega(z) = 2\,(1-|z|^2)\,\big[\Lambda_\omega(z)-1\big], & z\in\mathbb{D}, \\ \Phi_\omega(z) = 0, \qquad z\in\mathbb{T}. \end{cases}$$

It follows from Green's formula and Lemma 2.5, as in [7], that the potential function has

$$\frac{\partial}{\partial n(z)}\Phi_\omega(z) = 0, \qquad z\in\mathbb{T}.$$

Then Φ_ω solves the over-determinated boundary value problem

$$\begin{cases} \Delta\Phi_\omega(z) = 2\,(1-|z|^2)\,\big[\Lambda_\omega(z)-1\big], & z\in\mathbb{D} \\ \Phi_\omega(z) = 0, \qquad z\in\mathbb{T}, \\ \dfrac{\partial}{\partial n(z)}\Phi_\omega(z) = 0, \qquad z\in\mathbb{T}. \end{cases}$$

Increasing the degree of the elliptic operator, we find that Φ_ω also solves the problem

$$\begin{cases} \Delta(1-|z|^2)^{-1}\Delta\Phi_\omega(z) = 2\Delta\Lambda_\omega(z)\geq 0, & z\in\mathbb{D} \\ \Phi_\omega(z) = 0, \qquad z\in\mathbb{T}, \\ \dfrac{\partial}{\partial n(z)}\Phi_\omega(z) = 0, \qquad z\in\mathbb{T}, \end{cases}$$

which has a unique solution. Then Φ_ω may be expressed in terms of the Green function Γ_1,

$$\Phi_\omega(z) = 2\int_{\mathbb{D}}\Gamma_1(z,w)\Delta\Lambda_\omega(w)\,d\Sigma(w)\geq 0, \qquad z\in\mathbb{D}.$$

Let us consider now that u is a subharmonic function which is C^2-smooth on $\overline{\mathbb{D}}$. Then, by applying Green's theorem, it follows that

$$\int_{\mathbb{D}}(\Lambda_\omega(z)-1)\,u(z)\,2\,(1-|z|^2)\,d\Sigma(z) = \int_{\mathbb{D}}\Phi_\omega(z)\Delta u(z)\,d\Sigma(z)\geq 0.$$

In the case when u is not smooth up to the boundary, we consider the dilated function $u_r(z)=u(rz)$, with $0<r<1$, for which the above inequality holds. It follows from the growth bounds of u and Λ, and Lebesguet́s dominated convergence theorem, that we may let $r\to 1^-$ to conclude that

$$\int_{\mathbb{D}}(\Lambda_\omega(z)-1)\,u(z)\,2\,(1-|z|^2)\,d\Sigma(z)\geq 0.$$

The proof is complete. $\square$

3 The proof of Theorem 1.1

We realized back in the introduction that, in order to obtain Theorem 1.1, all we need to do is show that

$$K_\omega(z,w) \neq 0, \qquad (z,w) \in \big(\bar{\mathbb{D}} \times \mathbb{D}\big) \cup \big(\mathbb{D} \times \bar{\mathbb{D}}\big),$$

provided that ω is a C^∞-smooth and positive weight function on $\bar{\mathbb{D}}$, with the property that

$$z \mapsto \log \frac{\omega(z)}{1-|z|^2}$$

is subharmonic on $\mathbb{D}$. After all, Theorem 2.4 guarantees that the weighted Bergman kernel is C^∞-smooth up to the boundary. It is easy to verify that the above subharmonicity requirement is the same as the condition on the curvature form in the statement of Theorem 1.1.

The proof splits naturally into two parts.

PROPOSITION 3.1. *Under the above conditions on ω,*

$$K_\omega(z,w) \neq 0, \qquad (z,w) \in \mathbb{D} \times \mathbb{D}.$$

Proof. We observe first that for any Mœbius map ϕ preserving the disk $\mathbb{D}$, we have

$$K_\omega(\phi(z), \phi(w)) = K_{\omega_\phi}(z,w), \qquad z,w \in \mathbb{D},$$

where

$$\omega_\phi(z) = |\phi'(z)|^2\, \omega \circ \phi(z), \qquad z \in \mathbb{D}.$$

We claim that ω_ϕ is a weight of the same type as ω. In fact, the function

$$z \mapsto \log\left(\frac{\omega_\phi(z)}{1-|z|^2}\right)$$

is subharmonic on $\mathbb{D}$ if, and only if, the function

$$z \mapsto \log\left(\frac{\omega(z)}{1-|\phi^{-1}(z)|^2}\right)$$

is subharmonic on $\mathbb{D}$ as well. Then, if we consider a Mœbius map

$$\phi^{-1}(z) = \gamma \frac{z-\zeta}{1-z\bar{\zeta}}, \qquad z,\zeta \in \mathbb{D}, \quad |\gamma| = 1,$$

we find that

$$\log\left(\frac{\omega(z)}{1-|\phi^{-1}(z)|^2}\right) = \log\left(\frac{\omega(z)}{1-|z|^2}\right) + \log\left(\frac{|1-z\bar{\zeta}|^2}{1-|\zeta|^2}\right).$$

Thus, the function

$$z \mapsto \log\left(\frac{\omega_\phi(z)}{1-|z|^2}\right)$$

is subharmonic on $\mathbb{D}$ if, and only if,

$$z \mapsto \log\left(\frac{\omega(z)}{1-|z|^2}\right)$$

is subharmonic on $\mathbb{D}$ as well. It follows that it is enough to specialize to $w = 0$:

$$K_\omega(z,0) \neq 0, \qquad z \in \mathbb{D}.$$

We note that, by the reproducing property of the kernel function, $K_\omega(0,0) = \|K_\omega(z,0)\|_\omega^2 > 0$. We introduce the extremal function L, given by

$$L(z) = (K_\omega(0,0))^{-\frac{1}{2}} K_\omega(z,0), \qquad z \in \mathbb{D},$$

which solves the problem

$$\sup\{\operatorname{Re} f(0) : \|f\|_\omega \leq 1\}.$$

By Theorem 2.4, the function L is C^∞-smooth on $\overline{\mathbb{D}}$.

We argue by contradiction. So, we assume that there exists a $\lambda \in \mathbb{D}$ such that $K_\omega(\lambda,0) = 0$; then $L(\lambda) = 0$. Consider then the function

$$\tilde{L}(z) = L(z)/\varphi_\lambda(z), \qquad z \in \mathbb{D}$$

where φ_λ is the canonical divisor of $\{\lambda\}$ in the space $\mathcal{A}_1^2(\mathbb{D})$. We should point out that, due to the smoothness of φ_λ and the fact that it doesn't have any extraneous zeros on $\overline{\mathbb{D}}$, the function $\tilde{L}$ is also C^∞-smooth on $\overline{\mathbb{D}}$.

Let u be the function given by

$$u(z) = \frac{\omega(z)}{2(1-|z|^2)} |\tilde{L}(z)|^2, \qquad z \in \overline{\mathbb{D}}.$$

It follows from the hypothesis on the weight that $\log u$ is a subharmonic function in $\mathbb{D}$, so that in particular, u is subharmonic as well and it has the growth bound

$$|u(z)| = O\left(\frac{1}{1-|z|}\right), \qquad \text{as} \quad |z| \to 1^-.$$

It follows from Lemma 2.3 that

$$\begin{aligned}
\|\tilde{L}\|_\omega^2 &= \int_{\mathbb{D}} |\tilde{L}(z)|^2 \, \omega(z) \, d\Sigma(z) \\
&= \int_{\mathbb{D}} u(z) \, 2\,(1-|z|^2) \, d\Sigma(z) \leq \int_{\mathbb{D}} |\varphi_\lambda(z)|^2 \, u(z) \, 2\,(1-|z|^2) \, d\Sigma(z) \\
&\qquad\qquad = \int_{\mathbb{D}} |L(z)|^2 \, \omega(z) \, d\Sigma(z) = \|L\|_\omega^2 = 1.
\end{aligned}$$

On the other hand, it follows from equation (2.1) that $\varphi_\lambda(0) < 1$ and so $\tilde{L}(0) > L(0)$, which violates the extremal property of $L(z)$. This is the desired contradiction. Hence, the function L does not have zeroes in $\mathbb{D}$. $\square$

PROPOSITION 3.2. *Under the above conditions on ω,*

$$K_\omega(z,w) \neq 0, \qquad (z,w) \in \big(\mathbb{T}\times\mathbb{D}\big) \cup \big(\mathbb{D}\times\mathbb{T}\big).$$

Proof. Note that, by the same argument used in the proof of the Proposition 3.1 and the fact that

$$K_\omega(z,w) = \overline{K}_\omega(w,z), \qquad z,w \in \overline{\mathbb{D}},$$

it is enough to prove that

$$K_\omega(z,0) \neq 0, \qquad z \in \mathbb{T}.$$

We argue by contradiction. So, we shall assume that there exists $\lambda \in \mathbb{T}$ such that $K_\omega(\lambda,0) = 0$. Then, due to the smoothness of $z \mapsto K_\omega(z,0)$ on $\bar{\mathbb{D}}$, as provided by Theorem 2.4, we find that

$$|K_\omega(z,0)| = O\big(|z-\lambda|\big), \qquad \text{as} \quad \mathbb{D} \ni z \to \lambda.$$

Let $0 \le r < 1$ and define the function $f_r \in \mathcal{A}^2_1(\mathbb{D})$, given by

$$f_r(z) = \frac{K_1(z, r\lambda)}{\sqrt{K_1(r\lambda, r\lambda)}} = \frac{(1-r^2)^{\frac{3}{2}}}{(1-r\overline{\lambda}z)^3}, \quad z \in \mathbb{D},$$

where K_1 is the reproducing kernel for the space $\mathcal{A}^2_1(\mathbb{D})$. It follows that

$$\|f_r\|_1 = 1, \qquad 0 \le r < 1,$$

where $\|\cdot\|_1$ is the norm in $\mathcal{A}^2_1(\mathbb{D})$, as defined back in Section 2. Furthermore, $|f_r|^2$ is bounded on $\mathbb{D}$ for each $0 \le r < 1$. We now consider the function

$$R_\omega(z) = \frac{|K_\omega(z,0)|^2}{K_\omega(0,0)}\omega(z), \quad z \in \mathbb{D}.$$

It follows that

$$R_\omega(z) = O\big(|z-\lambda|^2\big) \qquad \text{as} \quad \mathbb{D} \ni z \to \lambda.$$

Then there exists a positive constant M such that

$$R_\omega(z) \le M\,|z-\lambda|^2, \qquad z \in \mathbb{D}.$$

It follows from Lemma 2.6 and the inequality

$$r\,|z-\lambda| \le \big|1 - r\,\overline{\lambda}z\big|, \qquad z \in \mathbb{D}, \quad 0 < r < 1,$$

that

$$\begin{aligned} 1 = \int_{\mathbb{D}} |f_r(z)|^2\, 2\,(1-|z|^2)\, d\Sigma(z) &\le \int_{\mathbb{D}} \Lambda_\omega(z) |f_r(z)|^2\, 2\,(1-|z|^2)\, d\Sigma(z) \\ = \int_{\mathbb{D}} R_\omega(z)\, |f_r(z)|^2\, d\Sigma(z) &\le M \int_{\mathbb{D}} |z-\lambda|^2\, |f_r(z)|^2\, d\Sigma(z) \\ &\le M\,\frac{(1-r^2)^3}{r^2} \int_{\mathbb{D}} \frac{1}{|1-r\overline{\lambda}z|^4} d\Sigma(z) = M\frac{1-r^2}{r^2}, \end{aligned}$$

which is a contradiction for r sufficiently close to 1. $\square$

4 The proof of Theorem 1.2

In this section, we construct, for a fixed $\alpha \geq \alpha_0 = 1.04$, an explicit example of a function $\mu \in C^\infty(\mathbb{D})$, with the associated 2-form $\boldsymbol{\mu} = \mu d\Sigma$, such that

$$\boldsymbol{\mu}(z) + \frac{\alpha}{2}\,\mathbf{K}_{\mathbb{H}}(z) \leq 0, \qquad z \in \mathbb{D}, \tag{4.1}$$

for which the optimization problem (OP) fails to have a smooth-positive solution ω_0 on $\mathbb{D}$, as the weighted Bergman kernel $K_{\omega_1}(\cdot, 0)$ has an extraneous zero in $\mathbb{D}$. Here, ω_1 is associated with μ as in the introduction:

$$\log \omega_1(z) = -\int_{\mathbb{D}} \log \left| \frac{z - w}{1 - z\bar{w}} \right| \mu(w)\, d\Sigma(w), \qquad z \in \mathbb{D}. \tag{4.2}$$

The choice of $\boldsymbol{\mu}$. We first consider the extremal case for the inequality (4.1),

$$\boldsymbol{\mu}_{\mathbb{H}}(z) = -\frac{\alpha}{2}\,\mathbf{K}_{\mathbb{H}}(z) = \frac{2\,\alpha\, d\Sigma(z)}{(1 - |z|^2)^2}, \quad z \in \mathbb{D}.$$

In this case we can compute explicitly the weight ω_0, which solves the optimization problem (OP),

$$\omega_0(z) = (1 - |z|^2)^\alpha, \qquad z \in \mathbb{D}.$$

Following the same line of thought as in our toy example from back in the introduction, we then consider the data function

$$\mu(z) = \frac{2\,\alpha}{(1 - |z|^2)^2} - \sum_k \rho_k\, \delta_{a_k}(z), \qquad z \in \mathbb{D}, \tag{4.3}$$

where $A = \{a_k\}_k$ is a finite collection of points in $\mathbb{D}$, $\{\rho_k\}_k$ is a convenient sequence of positive constants, and δ_{a_k} stands for the Dirac delta function, concentrated at the point $a_k \in \mathbb{D}$. The 2-form $\boldsymbol{\mu} = \mu d\Sigma$ meets the inequality (4.1), but μ is very rough at the points of A and hence it does not satisfy the C^∞-smoothness requirement. However, it is easy to approximate a point-mass by a sequence of positive C^∞-smooth functions. Moreover, we may replace the function

$$\nu(z) = \frac{2\,\alpha}{(1 - |z|^2)^2}$$

by a slight dilation,

$$\nu_r(z) = \frac{2\,r^2\,\alpha}{(1 - r^2\,|z|^2)^2}$$

with r, $0 < r < 1$, close to 1. This means that if

$$K_{\omega_1}(z, w) \neq 0, \qquad z, w \in \mathbb{D}, \tag{4.4}$$

holds for smooth indata μ, with μ and ω_1 connected via (4.2), then it also holds for rough indata of the above type, modulo some slight modifications. To explain these modifications, we note that

$$\int_{\mathbb{D}} \log\left|\frac{z-w}{1-z\bar{w}}\right| \nu_r(w)\, d\Sigma(w) = \alpha \log \frac{1-r^2}{1-r^2|z|^2}, \qquad z \in \mathbb{D}.$$

Let the weight $\omega_{1,r}$ be defined by the formula (4.2), where ω_1 is replaced by $\omega_{1,r}$, and μ is replaced by

$$\mu_r(z) = \frac{2\, r^2\, \alpha}{(1-r^2\,|z|^2)^2} - \sum_k \rho_k\, \delta_{a_k}(z), \qquad z \in \mathbb{D}.$$

We then calculate that

$$\omega_{1,r}(z) = (1-r^2)^{-\alpha}\big(1-r^2\,|z|^2\big)^{\alpha} \prod_k \left|\frac{z-a_k}{1-\bar{a}_k z}\right|^{\rho_k}, \qquad z \in \mathbb{D}.$$

As reproducing kernel functions have homogeneity index -1 in general, that is,

$$K_{t\omega}(z,w) = \frac{1}{t} K_\omega(z,w),$$

holds for arbitrary positive constant t, we see that

$$K_{\omega_{1,r}}(z,w) = (1-r^2)^{\alpha}\, K_{\omega_{2,r}}(z,w),$$

where $\omega_{2,r}$ is the weight

$$\omega_{2,r}(z) = \big(1-r^2\,|z|^2\big)^{\alpha} \prod_k \left|\frac{z-a_k}{1-\bar{a}_k z}\right|^{\rho_k}, \qquad z \in \mathbb{D}.$$

If (4.4) holds for $\omega_1 = \omega_{1,r}$, then it also holds for $\omega_1 = \omega_{2,r}$, and vice versa. As $r \to 1^-$, the weight $\omega_{2,r}$ tends to

$$\omega_2(z) = \big(1-|z|^2\big)^{\alpha} \prod_k \left|\frac{z-a_k}{1-\bar{a}_k z}\right|^{\rho_k}, \qquad z \in \mathbb{D}, \tag{4.5}$$

and if the reproducing kernel functions for the weights $\omega_{2,r}$ are all zero-free in $\mathbb{D}^2$, then so is the reproducing kernel for the weight ω_2, by a limit process argument. We shall prove that with appropriate choices of the configuration of the points $A = \{a_k\}_k$ as well as of the positive parameters ρ_k, the reproducing kernel function for ω_2 will have zeros in $\mathbb{D}^2$. This then shows that also with smooth data, we must have zeros in the associated reproducing kernel function.

We should note that ω_2 satisfies

$$\Delta \log \omega_2(z) = -\frac{1}{2}\,\mu(z), \qquad z \in \mathbb{D},$$

where μ is given by (4.3). This means that ω_2 is the weight we where looking for.

For computational reasons, we shall only consider ρ_k such that $\rho_k/2$ is a positive integer.

Relations between kernel functions. A closed subspace I of $\mathcal{A}^2_\alpha(\mathbb{D})$ is called invariant if it is invariant under the multiplication by the identity function, more precisely, if $zf \in I$ whenever $f \in I$. For a sequence $A = \{a_1, a_2, \ldots\}$ of points on $\mathbb{D}$, the subspace I_A consisting of all functions in $A^2_\alpha(\mathbb{D})$ whose zero sets contain A, counting multiplicities, is an invariant subspace. Such a subspace I_A is called *zero-based invariant subspace*. The reproducing kernel function for the invariant subspace I_A is as usual defined by the formula

$$K^\alpha_A(z,w) = \sum_{n=0}^{+\infty} e_n(z)\,\bar{e}_n(w), \qquad z,w \in \mathbb{D},$$

where the functions $e_1(z)$, $e_2(z)$, $e_3(z), \ldots$ form an orthonormal basis for I_A. It has the reproducing property

$$f(z) = \int_{\mathbb{D}} K^\alpha_A(z,w)\, f(w)\, (\alpha+1)\big(1-|w|^2\big)^\alpha d\Sigma(w), \qquad z \in \mathbb{D},$$

for $f \in I_A$. For a finite subset $A = \{a_k\}_k$ of $\mathbb{D}$, the associated (finite) Blaschke product is the function

$$B_A(z) = \prod_k \frac{z - a_k}{1 - \bar{a}_k z}, \qquad z \in \mathbb{D}.$$

We consider the following two weights:

$$\omega_\alpha(z) = (\alpha+1)\big(1-|z|^2\big)^\alpha, \qquad \omega_{\alpha,A}(z) = (\alpha+1)\big(1-|z|^2\big)^\alpha\,|B_A(z)|^2.$$

Note that $\omega_{\alpha,A}$ equals the weight ω_2 as defined by (4.5), with all the parameters ρ_k set equal to 2.

The following proposition is well known.

PROPOSITION 4.1. *We have the following identity of kernels:*

$$K^\alpha_A(z,w) = B_A(z)\,\bar{B}_A(w)\,K_{\omega_{\alpha,A}}(z,w), \qquad z,w \in \mathbb{D}.$$

In view of the above proposition, we need to look for extraneous zeros in the reproducing kernel function for I_A in order to get zeros of the kernel function for the weight $\omega_2 = \omega_{\alpha,A}$.

The kernel function for a zero-based invariant subspace. When the sequence A consist of a finite number of distinct points, the kernel function K^α_A for I_A may be obtained by means of the well-known iterative formula (see [9])

$$K^\alpha_\emptyset(z,w) = \frac{1}{(1-z\overline{w})^{(\alpha+2)}}, \qquad z,w \in \mathbb{D} \tag{4.6}$$

$$K^\alpha_{A\cup\{\lambda\}}(z,w) = K^\alpha_A(z,w) - \frac{K^\alpha_A(z,\lambda)K^\alpha_A(\lambda,w)}{K^\alpha_A(\lambda,\lambda)}, \qquad \lambda \notin A. \tag{4.7}$$

The first step of this iteration is to apply the formula (4.7) to the case $A = \emptyset$ and $\lambda \in \mathbb{D}$, to get

$$K_\lambda^\alpha(z,w) = \frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{(1-|\lambda|^2)^{\alpha+2}}{(1-z\bar{\lambda})^{\alpha+2}(1-\lambda\bar{w})^{\alpha+2}}, \qquad z,w \in \mathbb{D}.$$

In the case where the zero set A contains repeated points, i.e. zeros with high multiplicity, the kernel function K_A^α for I_A can be computed via an iterative formula, similar to (4.7), but involving the derivatives of the kernel. Namely, if we assume A has no multiple points, then

$$K_\emptyset^\alpha(z,w) = \frac{1}{(1-z\overline{w})^{(\alpha+2)}}, \qquad z,w \in \mathbb{D},$$

$$K_{A\cup\{\lambda\}}^\alpha(z,w) = K_A^\alpha(z,w) - \frac{K_A^\alpha(z,\lambda)K_A^\alpha(\lambda,w)}{K_A^\alpha(\lambda,\lambda)}, \qquad \lambda \notin A,$$

$$K_{A\cup\{\xi\}}^\alpha(z,w) = K_A^\alpha(z,w) - \frac{\partial_z K_A^\alpha(z,w)\big|_{z=\xi}\ \bar{\partial}_w K_A^\alpha(z,w)\big|_{w=\xi}}{\partial_z\bar{\partial}_w K_A^\alpha(z,w)\big|_{z=\xi,\, w=\xi}}, \qquad \xi \in A,$$

where

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \bar{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right), \qquad z = x + \mathrm{i}y.$$

When A consist of n copies of the point $a \in \mathbb{D}$, the reproducing kernel function K_A^α for the subspace I_A is

$$K_A^\alpha(z,w) = \frac{1}{(1-z\bar{w})^{\alpha+2}} - \frac{(1-|a|^2)^{\alpha+2}}{(1-z\bar{a})^{\alpha+2}(1-a\bar{w})^{\alpha+2}} \sum_{j=0}^{n-1} C_{\alpha,j}\left(\frac{z-a}{1-z\bar{a}}\right)^j \left(\frac{\bar{w}-\bar{a}}{1-a\bar{w}}\right)^j,$$

where

$$C_{\alpha,j} = \frac{\Gamma(\alpha+2+j)}{\Gamma(\alpha+2)j!}.$$

Our computational work suggest that whenever α is bigger than 3, it is possible to find extraneous zeros; for $\alpha \le 3$, however, it seems that such extraneous zeros do not occur.

Computational implementation. The recursive formula for (4.7) is somewhat inconvenient in computational applications. Instead, we use the fact that the kernel function with $w = 0$ may be expressed as

$$K_A^\alpha(z,0) = 1 - \sum_j \frac{c_j}{(1-z\bar{a}_j)^{\alpha+2}}, \tag{4.8}$$

where the $\{c_j\}_j$ are certain complex coefficients, which depend on the sequence$A$ and on the parameter α. To determine them, we need to solve the linear system $K_A^\alpha(a_j, 0) = 0$, for every $a_j \in A$. This we rewrite in terms of a square matrix $\mathbf{M}$, whose entries are given by

$$M_{i,j} = \frac{1}{(1 - a_i\overline{a}_j)^{\alpha+2}}, \qquad a_i, a_j \in A;$$

the equation that determines the coefficients is the linear system

$$\mathbf{M}\,\mathbf{c} = \mathbf{1}, \tag{4.9}$$

where $\mathbf{c} = \{c_j\}_j$ is the coefficient vector in column form, and $\mathbf{1}$ is the column vector with the number 1 in all its coordinates.

Although the numerical computation of the kernel function K_A^α for several zeros via the linear system (4.9) is technically possible, we are to treat a rather badly scaled matrix. Numerically, the matrix $\mathbf{M}$ is nearly singular, and this leads to possibly large numerical error while solving the system (4.9). In addition, the kernel function expressed as expressed by (4.8) is quite sensitive to even small perturbations in the appearing coefficients already when the number of zeros in A is rather modest. Taken together, this forces us to work with higher precision than what is standard. We use 40 decimal digits of precision, instead of the standard 16. Most of the numerical tests were done in `MATLAB` 6.1. To work with forty decimal digits, we used the command `VPA` (variable precision arithmetic). We estimate the computational error in the approximated kernel function $K_A^\alpha(\cdot, 0)$ by comparing the computed values at the points of the given zero set $A = \{a_k\}_k$, where the function vanishes. If these values are sufficiently small, we may be certain that the negativity of $K_A^\alpha(1, 0)$ is a genuine phenomenon, provided that the negative value is substantially bigger than the numerically obtained values at the given collection of zeros.

The appearance of an extraneous zero. We first observe that if the elements of the zero set A are distributed symmetrically respect to the real axis, then the kernel function $K_A^\alpha(x, 0)$ is real-valued for real x. Since, trivially, $0 < K_A^\alpha(0, 0)$, and since the function $K_A^\alpha(z, 0)$ is continuous in the closed unit disk $\bar{\mathbb{D}}$, it follows from the Mean Value Theorem of Calculus that $K_A^\alpha(x_0, 0) = 0$ holds for some x_0, $0 < x_0 < 1$, provided that $K_A^\alpha(1, 0) < 0$.

After testing several patterns for the distribution of the zero set A (including, for instance, a multiple zero at a single point), we focused our computations on a configuration which yields extraneous zeros for α all the way down to 1.04. The same pattern seemed to emerge also when the computer was allowed to pick the configuration of the given zeros according to a so-called *genetic algorithm*. The configuration depends on the number n of points of A, as well as on two parameters θ and d, with $0 < \theta < \frac{1}{2}\pi$ and $1 < d < +\infty$. By the construction, n is an even number. The points of A are given by

$$\begin{aligned} a_k &= \exp\left\{3(\mathrm{i} - \theta)\, d^{k-n/2}\right\}, \qquad && k = 1, 2, \dots, n/2, \\ a_{k+\frac{n}{2}} &= \bar{a}_k, && k = 1, 2, \dots, n/2. \end{aligned}$$

Figure 1: Zero set A for $n = 80$ generated by the parameters $\theta = 0.15,\ d = 1.30$, for $\alpha = 1.50$. The indicated straight lines are tangents to the curves on which the zeros are located.

Figure 2: (top) Level curves of the kernel function $K_A^\alpha(\cdot, 0)$ for the zero set A generated by the parameter values: $\alpha = 3,\ n = 6,\ \theta = 0.51,\ d = 10$. (bottom) Values of the kernel function $K_A^\alpha(1, 0)$ for the zero set A generated by the parameters: $n = 1500,\ \theta = 0.033,\ d = 1.045$ and $1.035 \leq \alpha \leq 1.050$

In Figure 1, we plot the configuration of the given zero set in the complex plane for $n = 80$ zeros, with the parameter values $\alpha = 1.5$, $\theta = 0.15$, and $d = 1.3$.

In Figure 2 (top), we plot the level curves around the extraneous zero of the modulus of the kernel function for the zero set A generated by the parameter values $\alpha = 3$, $n = 6$, $\theta = 0.51$, and $d = 10$. In Figure 2 (bottom), we plot the kernel function $K_A^\alpha(1, 0)$, while varying α in the interval $[1.035, 1.050]$.

Inspiration for our numerical work was derived from [12], where Jakobsson obtained extraneous zeros for $\alpha \approx 1.40$.

In the table below (Table 1), we supply values of n, θ, and d, for which our computations indicated that the kernel function possesses extraneous zeros for a prescribed value of the parameter α.

α	n	θ	d
3	6	0.51	10
2.5	8	0.48	8
2	14	0.351	3
1.6	26	0.265	2.1
1.25	78	0.176	1.52
1.118	230	0.104	1.22
1.1072	272	0.092	1.183
1.097	340	0.07725	1.141
1.065	550	0.07	1.13
1.053	770	0.0556	1.09
1.046	944	0.0497	1.078
1.043	1090	0.0445	1.067
1.04	1500	0.033	1.045

Table 1.

Hinted analytical solution of the problem. The configuration of zeros in the above numerically-based counterexamples which works at least down to the parameter value $\alpha \approx 1.04$ suggests that analytically, we should "smear out" the zeros along the two lines; we may decide to work almost infinitesimally close to to the point 1, and by blowing up, we may assume that the domain is the upper half plane, where the origin plays the role of the point 1. We place hyperbolically equi-distributed smeared-out "zeros" along two half-lines emanating from the origin, symmetrically located with respect to the imaginary axis. In geometric terms, this means that we construct a new weight – the old one being $(\operatorname{Im} z)^\alpha$ – which is the same as the old weight in the region between the two half-lines, but is the old weight times the exponential of a constant times the angle to the nearest half-line in the remaining two regions near the real line. We suspect that the reproducing kernel for this new weight (with one argument fixed equal to i) has a zero somewhere along the imaginary axis for each fixed α, $1 < \alpha < +\infty$, provided the angle of the two half-lines to the real line are chosen appropriately, and the constant that regulates the density of the "zeros" is appropriate as well. However, the actual implementation of this scheme is not easy, because it is generally a hard problem to calculate reproducing kernel functions for weights that do not exhibit strong symmetry properties. We would like to be able to return to this problem.

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